HELICOPTER BLADE DROOP AS AN APPLICATION OF SIMPLE BEAM THEORY

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PACS : 46.25.-y

Keywords : Helicopter blade Cantilever beam Centrifugal force

Abstract

It is a commonly observed fact that the blades of a helicopter droop under their own weight when the aircraft is static but become almost straight when they start spinning. This phenomenon is commonly attributed to "rotational stiffness". In this short article we present a simple calculation for the shape of the blades of a helicopter in static and rotating condition. We show that the deflection or droop of the tip of the blade reduces by a factor of about 30 when it changes from static to full speed rotation. This calculation is at the undergraduate level and can complement a course in differential equations, elementary statics or elasticity theory.

Introduction

Beam theory is a core component of undergraduate and graduate courses in statics and elasticity [1-5]. It is also an optional component of a course on differential equations, in which case it provides a natural example of a high order ordinary differential equation (ODE) boundary value problem. While problems like doubly clamped and cantilever beams with various kinds of loading are quite common, a non-trivial situation which lends itself to a beam theoretic model is that of a helicopter blade. It is a common observation that the helicopter blades droop when they are static and become nearly straight when they start spinning. This is generally attributed to "rotational stiffness" [6]; here we actually calculate the shapes in the static and rotating situations. A prior treatment of helicopter blade as a cantilever beam can be found in the book by Johnson [7]; he uses the cantilever model to study the dynamics of the flapping blades but does not comment on its static shape. A numerical approach which models the blades as beams for a dynamical analysis can be found in the work of Mayo et. al. [8]. The following simple calculation for the blade shape however appears to be novel in the literature.

The Shape of the static blade

We model each blade of the main fan as a cantilever of uniform cross-section which is clamped at the centre of the fan (x=0) and free at the far end (x=L) where L is the beam length. For a typical helicopter [9], the blade length is 7 m and the cross-section approximately rectangular with breadth 0.8 m and thickness 0.004 m (4 mm). The cross-sectional moment of inertia is 7.5x10^{-6} m^4, the elastic modulus of the material is 10^{11} N/m^2 (100 GPa) and the density is 3000 kg/m^3. The undeformed shape of the blade in the absence of all forces is perfectly horizontal.

When the helicopter is dead, the blade hangs under its own weight. Under this condition, the differential equation satisfied by the blade is

$$EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -\rho_1 g \quad , \tag{1}$$

where x measures the radially outward direction, y is the vertical deviation from the undeflected state, EI is the flexural rigidity, ρ_1 the mass per unit length of the beam and there are the standard cantilever boundary conditions (BCs)

$y(0)=y'(0)=0$ at the centre and $y''(L)=y'''(L)=0$. This problem may be solved easily [10] and we find

$$y=-\frac{\rho_1 g}{EI}\left(\frac{1}{24L}x^4-\frac{1}{6}x^3+\frac{L}{4}x^2\right) \quad . \tag{2}$$

Substituting the numerical values, EI evaluates to 750000 Nm^2, and $\rho_1 g$ to 941 N/m. Using the above solution, the deflection of the tip turns out to be 37.65 cm. It would have made more practical sense to report it as 38 cm but four place accuracy will be required as a comparison benchmark when we solve the rotating problem numerically.

The Force balance on the rotating blade

When the helicopter is live, the blade rotates with angular velocity ω about a vertical axis. We take a typical ω to be 35 rad/s (340 rpm). We perform the analysis in a reference frame which rotates with the blade. In this frame, the rotation gives rise to a purely horizontal (radial) centrifugal force $m\omega^2 x$ on each mass element m of the blade. This force acts in addition to the weight. We now formulate the equation for the shape of the blade in the presence of the rotation.

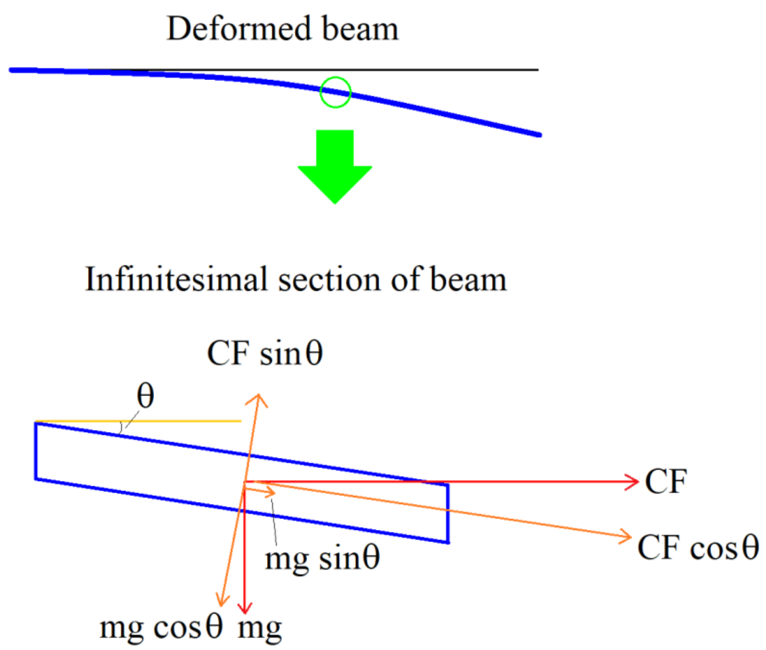

Figure 1 : Force balance on the beam. CF refers to the centrifugal force.

The infinitesimal section of blade shown in Fig. 1 is inclined at angle θ to the horizontal, where we take θ to be finite rather than small – the limit will be taken only at the end. The length of the section is dx/cosθ and its mass is ρ_1 times that where ρ_1 is the mass per unit length. The forces acting on it are gravity, which is vertical and centrifugal force which is horizontal. Gravity splits into a cosθ component transverse to the element and a sinθ component parallel to the element. The former produces bending while the latter contributes to axial stress and is rejected. Centrifugal force splits into a sinθ bender and a cosθ axial stressor which latter is neglected from consideration. The centrifugal force on the element is (dm)ω^2x. Thus, the bending forces are –(dm)$g$cosθ and (dm)ω^2xsinθ. Now, we take the limit θ is very small. The infinitesimal mass becomes ρ_1dx, the bending component of gravity becomes –(ρ_1dx)g and the bending component of centrifugal force becomes (ρ_1dx)$\omega^2x\theta$. But, dy/dx=–tanθ (the negative because θ is clockwise positive), and for small angles tanθ also tends to θ, so θ is actually –y' and the centrifugal bending term at location x is –(ρ_1dx)ω^2xy'. Finally, to get the force per unit length, we factor out dx from the force on the element and write the bending equation as

$$EIy^{(4)} = -\rho_1 g - \rho_1\omega^2 xy' \quad \text{, or} \tag{3a}$$

$$EIy^{(4)} + \rho_1\omega^2 xy' = -\rho_1 g \quad . \tag{3b}$$

The associated boundary conditions (BC) are the same as the ones for the usual cantilever, so we have $y(0)=y'(0)=y''(L)=y'''(L)=0$.

Solution and discussion

We use a finite differences solution scheme to solve (3). The details of this scheme are presented in the Appendix. We vary the level N of spatial discretization by using the static cantilever (ω=0) as benchmark, where we compare the numerical result with the analytical value of 37.65 cm. Maximum accuracy of solution for the static cantilever was obtained at $N = 3000$, where the static deflection evaluates to 37.54 cm, amounting to an accuracy level of about 3 in 1000. In Fig. 2 we present a comparison between the shapes of the static and rotating blades. The diagram is drawn to scale, and a schematic representation of the helicopter has been added. The deflection of the tip evaluates to 1.17 cm.

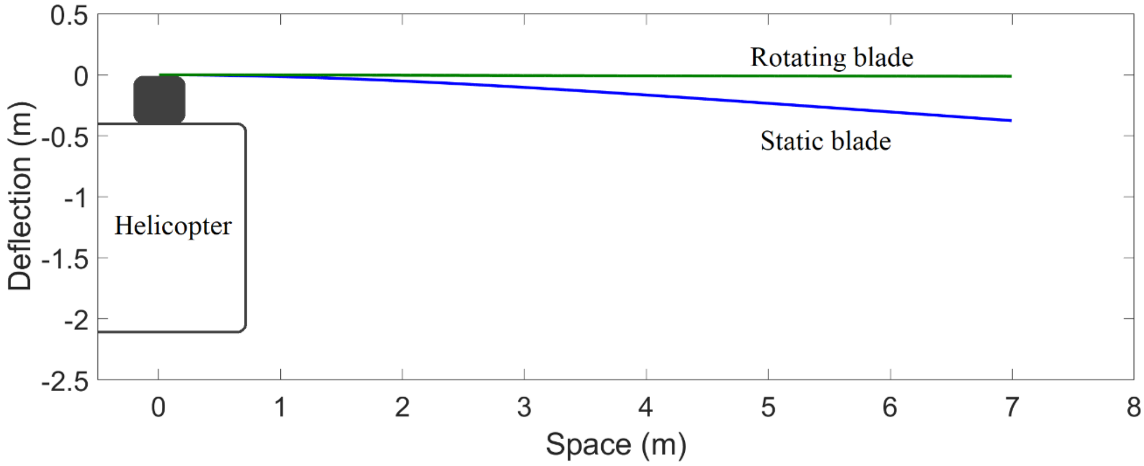

Figure 2 : Deflection of the beam as per the numerical solution. Comparison of static and rotating beam is presented.

We now analyse certain features of our solution. The most striking feature is that the deflection of the tip has reduced by a factor of 30. This is consistent with what we know from observation – that the blades of a helicopter droop when static and appear almost straight when running. It is also logical – centrifugal force provides a load opposite to the direction of gravity so it should raise the beam rather than lowering it. The force is also many times stronger than gravity – at the midpoint of the blade, $\omega^2 x$=4500 or so, while g is merely 10 ! Very crudely, the average slope of the beam is about 1/600 (1.17 cm drop in 700 cm run) and the average centrifugal force times the average slope is certainly of the same order of magnitude as gravity. Hence the numerical solution is very plausible.

In Fig. 3 we enlarge the rotating blade to examine the shape in detail. The shape appears physically realistic (for example it does not have any region with positive slope or wiggles) and reassures us that the numerical results are not implausible.

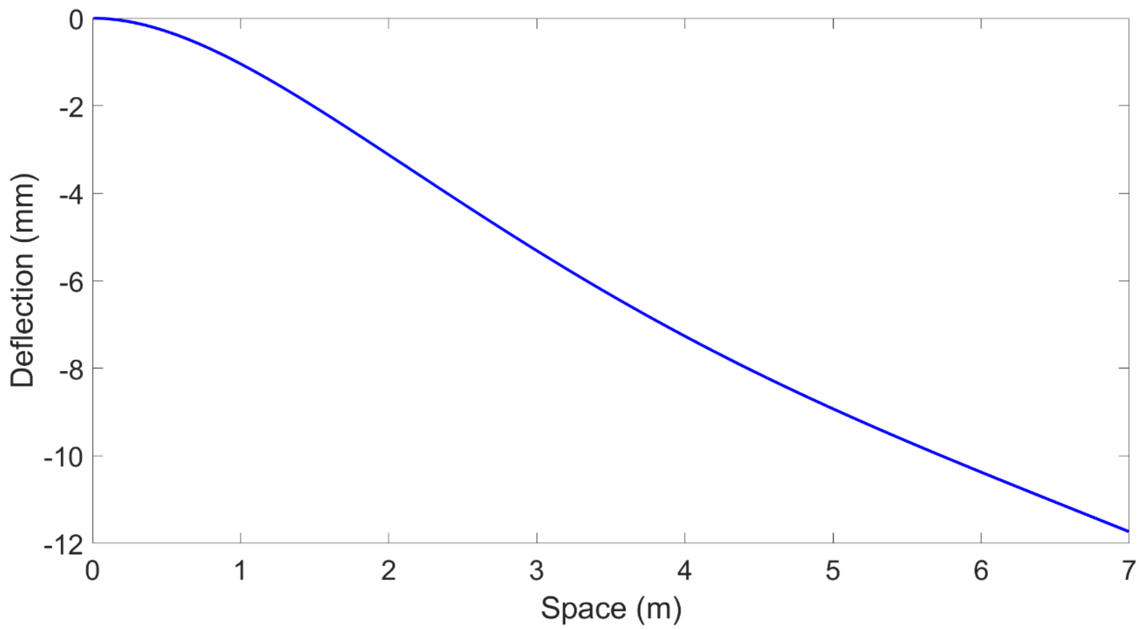

Figure 3 : Deflection of rotating beam as per numerical solution.

In view of this encouraging observation, we now go into the analysis of (3b) in a more systematic manner. We rewrite it as

$$L^4 y^{(4)} + \frac{\rho_1 \omega^2 L^4}{EI} x y' = -\frac{\rho_1 L^4 g}{EI} \quad , \tag{4}$$

which enables the identification of two dimensionless parameters $\kappa=\rho_1\omega^2L^4/EI$ and $\lambda=\rho_1L^3g/EI$. Taking their ratio gives the primary parameter driving the shape, which is $\alpha=\omega^2L/g$ i.e. the ratio of centrifugal force to gravity. A second dimensionless parameter is the ratio of the tip deflection in the presence of rotation to that in the absence of the rotation. We will call this ratio ν. In Fig. 4 we show a plot of ν vs. α as the latter is increased from zero (when ν=1) to 1200 (when ν=0.0241). We note that for the helicopter under consideration, ν=850 approximately.

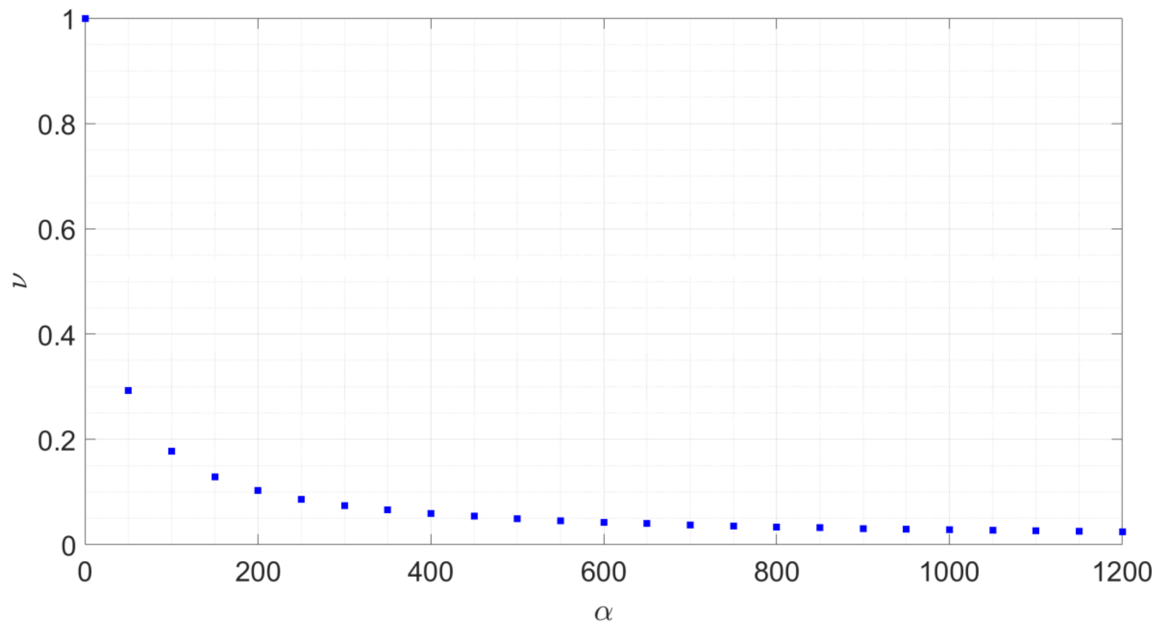

Figure 4 : The dimensionless parameter ν (ratio of rotating deflection to static deflection) vs the parameter α (ratio of centrifugal to gravitational acceleration).

The plot seems to show a power law decrease, so we now present a log-log plot. Recall that the logarithm is to the base 10.

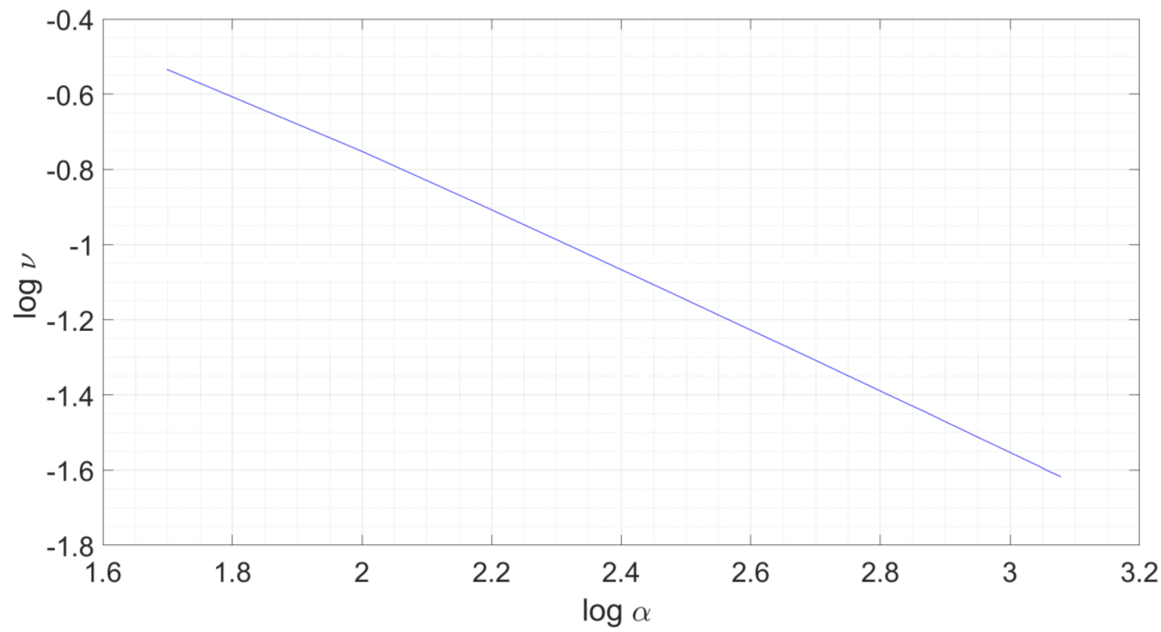

Figure 5 : Same as Fig. 4 but in log-log scale.

From this plot it appears that at high α, v varies approximately as $\alpha^{-0.8}$. A heuristic argument can explain the power law behaviour here. If we approximate the shape of the beam as a straight line, then the average of the centrifugal acceleration is $\omega^2 L/2$ and its vertical component is $\omega^2 L\theta/2$ where θ is the deflection angle. Equating this to the gravitational acceleration we have $\omega^2 L\theta/2=g$. The dimensionless deflection of the tip is θ so we immediately have θ varying as $(g/\omega^2 L)^{-1}$ i.e. v varies as α^{-1}.

The nearly flat nature of the plot in Fig. 4 at high α shows that the rotating blades will tend to appear straight for helicopters with parameter values different from our choices. Even at α=300, a very low value, the deflection is reduced to 7 percent of its static value. We also note that this non-dimensionalized analysis enables us to make predictions for the deflection when the helicopter is flying. In this case, the lift on the blade is the extra force. If we assume it to be uniform across the blade (the increasing translational speed of the blade away from the centre is at least partially compensated by the decreasing pitch as one moves radially outwards, so the assumption is reasonable – in any case the model is quite approximate), its effect will be to replace $-\rho_1 g$ on the RHS of (3b) by the lift per unit length. This will have positive sign since it is directed upwards. Its approximate magnitude will be 5 times higher than gravity, since a typical helicopter weighs 15 to 30 times more than a blade and the helicopter has 3 to 5 blades. The value of α in this case will be about 200, for which we get a v of 0.1 – the rise of the tip will still be 10 times lower than the droop when the helicopter is dead and the blade will appear very nearly straight when seen with the naked eye.

Concluding remarks

In this Article we have explained the straightening of rotating helicopter blades at a level which is easily accessible to the second year undergraduate student. It can complement a course in statics, as well as one in differential equations (in which context I had devised this problem). An excellent demonstration of the straightening of helicopter blades as they are spun up us found in a video of startup of Boeing Apache helicopter uploaded on Youtube by user "Michael Miller". The section of the video relevant for our purposes is reproduced at the following link : www.shayak.in/Shayakpapers/DELTA/T068_V1.mp4.

A more detailed analysis of the phenomenon can try to account for the –0.8 power law in v vs. α and can also predict corrections to the shape arising from the variations of EI along the blade, presence of nonlinearities and other phenomena. This however would take us outside the scope of the undergraduate classroom, and we leave it for a future study.

Appendix : Finite differences solution scheme

Equation (3b) is not solvable exactly, so we must use a numerical procedure. Although it is possible to solve this using Mathematica as a black box, we present a brief exposition of the numerical method for the benefit of those who don't possess this software package (like myself). Since the problem has boundary conditions specified rather than initial conditions (i.e. the data accompanying the differential equation is given at two different values of the independent variable instead of the same value of the independent variable), a direct method such as Euler's method or Runge Kutta method is ineffective. The most basic method is that of finite differences, also called the matrix method. Here we present significant details of the method so that an undergraduate student with adequate understanding of linear algebra but no experience with numerical methods can easily follow the discussion. Yet more details may be found in Reference [11].

In the preliminary step, the interval [0, L] is discretized into N significant points or lattice sites and two boundary points on each side (four extra points are needed for a problem with fourth derivative). Since $x = 0$ corresponds to site #–1 and $x = L$ to site #N+2, the step size h thus becomes $L/(N+3)$. The discrete second derivative is a well known formula

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad , \tag{A1}$$

and the third derivative is

$$y'''_i = \frac{1}{h}\left(y''_i - y''_{i-1}\right) \quad \text{, or} \tag{A2a}$$

$$y'''_i = \frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^3} \quad . \tag{A2b}$$

in the left side form and

$$y'''_i = \frac{y_{i+2} - 3y_{i+1} + 3y_i - y_{i-1}}{h^3} \quad , \tag{A3}$$

in the right form. The centred fourth derivative is

$$y^{(4)}{}_i = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} \quad . \tag{A4}$$

The raw fourth derivative matrix $\mathbf{S}_+$ contains the formula (A4) in every row, arranged in a stepwise manner. This matrix is not square, having four more columns than rows (this is indicated by the subscript "+"; the square form of the fourth derivative will be denoted by **S**). The discrete fourth derivative is $\mathbf{y}^{(4)} = \mathbf{S}_+\mathbf{y}_+$ and we now have to impose the boundary conditions (BC) to convert the non-square $\mathbf{S}_+$ to a square matrix **S**. The product $\mathbf{S}_+\mathbf{y}_+$ is shown below, with N taken as 4 for conceptual clarity and ease of display :

$$\mathbf{y}^{(4)} = \mathbf{S}_+\mathbf{y}_+ = \frac{1}{h^4}\begin{bmatrix} 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \end{bmatrix}\begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} \quad . \tag{A5}$$

The first BC is of course $y_{-1} = 0$. The next is $y'(0) = 0$. Taking a right side first derivative at site #–1; this yields $y'_{-1} = (y_0 - y_{-1})/h$. Setting it equal to zero gives $y_0 = 0$ also. So y_{-1} and y_0 are both zero and when we impose these conditions, the first two columns of $\mathbf{S}_+$ go away with no modification.

The BCs at the free end are implemented as a second derivative and a right side third derivative both at site #N. This yields a pair of relations for y_{N+1} and y_{N+2} in terms of the significant sites. We have (again considering the case where N=4)

$$y_5 = -y_3 + 2y_4 \quad , \tag{A6a}$$

$$y_6 = -2y_3 + 3y_4 \quad . \tag{A6b}$$

It is noted that if the BC derivatives had been implemented differently, for example, a second derivative taken at site #N+1 instead of N, then the calculations would have been more complicated but the end result would have been the same in the limit $h \rightarrow 0$.

We substitute these results into (A5) excluding the first two columns of $\mathbf{S}_+$ and the first two elements of $\mathbf{y}_+$:

$$\mathbf{y}^{(4)} = \frac{1}{h^4}\begin{bmatrix} 6 & -4 & 1 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -y_3+2y_4 \\ -2y_3+3y_4 \end{bmatrix} . \quad \text{(A7)}$$

Expanding the last two rows of the matrix, we can see that (A7) is equivalent to

$$\mathbf{y}^{(4)} = \mathbf{Sy} \quad \text{, where} \quad \text{(A8a)}$$

$$\mathbf{S} = \frac{1}{h^4}\begin{bmatrix} 6 & -4 & 1 & 0 \\ -4 & 6 & 4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} . \quad \text{(A8b)}$$

To complete the formulation of the problem, we must take into account the $\rho_1\omega^2 xy'$ term. We express $\mathbf{y}'$ as $\mathbf{P}_+\mathbf{y}_+$ where, $\mathbf{P}_+$ is the raw first derivative matrix and $\mathbf{y}_+$ this time has only one boundary point over and above the significant points. Conversion of $\mathbf{P}_+$ to $\mathbf{P}$ is most easily done by considering the extra point y_0, whose value is already known to be zero. Then, the first derivative is

$$\mathbf{y}' = \mathbf{Py} \quad \text{, where} \quad \text{(A9a)}$$

$$\mathbf{P} = \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} . \quad \text{(A9b)}$$

The factor of x before y' is represented as a matrix whose $(i,i)^{th}$ element is x_i and all other elements are zero i.e. $\mathbf{X} = \mathrm{diag}\,(x_1\ x_2\ \ldots..\ x_N)$. We note that the numerical value of $\rho\omega^2/EI$, given our problem parameters, is 0.1568 SI Units. Thus this term is

$$0.1568xy' \rightarrow 0.1568\mathbf{XPy} \quad \text{, where} \quad \text{(A10a)}$$

$$
\mathbf{X} = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix} \quad . \qquad \text{(A10b)}
$$

The left hand side of the matrix representation of (3b) is now complete.

Finally, we have the right hand side. That is almost trivial :

$$
\mathbf{b} = -0.0012544\left[1; \quad 1; \quad .....; \quad 1\right] \quad , \qquad \text{(A11)}
$$

where the '1's repeat N times and again SI Units have been used. Putting all this together we have the matrix representation of the boundary value problem :

$$
\left[\mathbf{S} + 0.1568\mathbf{XP}\right]\mathbf{y} = \mathbf{b} \quad , \qquad \text{(A12)}
$$

which is what enters into the computer. As a check on our procedure, we will also solve for the shape of the beam in the absence of rotation, which satisfies the equation $\mathbf{Sy} = \mathbf{b}$.

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